Reply by Authors to W. C. Robison and J. R. Nelson

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THE data presented by Robison and Nelson in the preceding comment are one of six sets given by Nelson in Ref. 2 of the comment. We have examined all six sets hoping to apply the concept of generalized choking to them. However, the outlet pressure into which the mixing-tube flow discharged was, in all cases, atmospheric pressure. The initial stagnation pressure of the driven stream was also, in all cases, atmospheric pressure. The initial stagnation pressure of the driving stream did not exceed 5.5 atm. Under these conditions we cannot be sure that the outlet pressure was low enough to permit generalized choking to occur.

We have a few small indications that the methods of calculation derived for generalized choking may apply in some cases even when the back pressure is too high for the choking actually to occur; however, these indications are not well enough established for us to be ready to apply the methods of generalized choking to the data of Nelson. All of the six sets of data show a pressure rise in the downstream portion of the mixing tube. This pressure rise probably indicates that an extended shock (pseudo shock) is being formed in the driving stream. Because the mixing tube is short, the extended shock is not completed before the flow reaches the end of the tube. If the downstream pressure were lowered, this shock probably would be pulled out of the mixing tube; and soon after that we would expect generalized choking to occur. The evidence of choking is, of course, that further lowering of the downstream pressure produces no change upstream from the plane of choking.

Robison and Nelson, in their Fig. 2, show an interesting situation in which the driving stream, after entering the mixing tube, first expands and then contracts. The space remaining for the driven gas has the form of a converging-diverging nozzle, and the wall pressures in Fig. 1 of the comment indicate that the driven stream very well may become sonic at its narrowest cross section and supersonic as it flows downstream. We are not prepared to say whether the total flow in Fig. 2 of Robison and Nelson is choked or not. Even if we accept the flow of the driven stream as sonic at its narrowest cross section, it is still possible that a change in outlet pressure would cause a readjustment of the driving and driven streams and that this readjustment would change the area of the narrowest cross section of the driven stream. In such a case, downstream conditions and upstream conditions would not be decoupled, and the flow would not be truly choked.

The situation shown in Fig. 2 of the comment shows a substantial departure from one dimensionality; the driving gas has a strong divergence as it enters the mixing tube. Also, if we traverse the driving stream at its first cross-section maximum, we cross several interference fringes. Presumably, these fringes indicate that the density and the pressure at the axis of the driving stream are lower than at the edges, where the driving and driven streams are in contact. Nonuniformity in direction of flow and nonuniformity in pressure are of course ignored in our one-dimensional treatment. If these nonuniformities do not seriously modify the over-all picture, our model may still give useful results. At present, there is not much information establishing the limits of usefulness of the one-dimensional approach when applied to two interacting streams. However, the one-dimensional method has proved very useful for single streams, and we are more optimistic than Robison and Nelson about its usefulness for two or more streams.

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Comment on "Computer Analysis of Asymmetric Free Vibrations of Ring-Stiffened Orthotropic Shells of Revolution"

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IN a recent article, Cohen[1] described a computer analysis of free vibrations of thin shells. Among the results he obtained were natural frequencies and modes for axisymmetric vibration of a 60° spherical dome, a case which had been studied previously by Kalnins.[2] Cohen found an extra frequency between the second and third frequencies for a fixed-hinged edge given in Table 1 of Kalnins[2] paper. The purpose of this Note is to make some remarks about this frequency and mode and to show how it can be found with very little computation by asymptotic theory.

The basis for these remarks is a paper by the writer,[3] in which approximate frequency conditions and modes were derived for spherical domes under a variety of edge conditions. For a fixed-hinged edge, the approximate frequency condition† is

$$[dP_{n_3}(\cos\phi)/d\phi]_{\phi=\psi}(2S(\psi) + \Lambda^{-1/2}\cot\psi\{(\tfrac{1}{8}) \times [S(\psi) - C(\psi)] + [(\tfrac{7}{8}) - \nu][S(\psi) + C(\psi)]\}) = \Lambda^{-1/2}P_{n_3}(\cos\psi)\,[\lambda_3 - (1-\nu) - (1-\nu^2)\Omega^2][C(\psi) - S(\psi)] \quad (1)$$

and the modes are for $\phi \neq 0$

$$w = P_{n_3}(\cos\phi) - (\tfrac{1}{2})P_{n_3}(\cos\psi)(\sin\psi/\sin\phi)^{1/2} \times \{[S(\phi)/S(\psi)] + e^{b(\phi-\psi)}\} \quad (2)$$

$$u/(1+\nu) = [\lambda_3 - (1-\nu) - (1-\nu^2)\Omega^2]^{-1}\, dP_{n_3}(\cos\phi)/d\phi + (\Lambda^{-1/2}/2) \times P_{n_3}(\cos\psi)(\sin\psi/\sin\phi)^{1/2}\{-[C(\phi)/S(\psi)] + e^{b(\phi-\psi)}\} \quad (3)$$

and for $\phi \approx 0$

$$w = P_{n_3}(\cos\phi) - (\Lambda^{1/4}/2)(\tfrac{1}{2}\pi\sin\psi)^{1/4}P_{n_3}(\cos\psi) \times J_0(\Lambda^{1/2}\phi)/S(\psi) \quad (4)$$

$$u/(1+\nu) = [\lambda_3 - 1 + \nu - (1-\nu^2)\Omega^2]^{-1}dP_{n_3}(\cos\phi)/d\phi + (\Lambda^{-1/4}/2)(\tfrac{1}{2}\pi\sin\psi)^{1/2}P_{n_3}(\cos\psi)J_1(\Lambda^{1/2}\phi)/S(\psi) \quad (5)$$

Here, u and w are the meridional and normal (outward) displacements; ϕ is the latitude coordinate ($\phi = 0$ is the pole, and $\phi = \psi$ the edge); and ν is Poisson's ratio. Also, if ω is the frequency, E Young's modulus, h, ρ, and R the thickness, density, and radius of the sphere, then

$$\Omega^2 = \omega^2R^2\rho/E$$

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† This is not quite the same as the frequency condition stated in Eq. (54) of the paper.[3] In obtaining the latter form, an additional approximation $S(\psi) = O(\Lambda^{-1/2})$ was introduced into the previous condition. That approximation is accurate only for the bending frequencies. Here we deal with membrane frequencies and thus must use the frequency condition in its pristine form (1).

$$\epsilon^2 = h^2/[12R^2(1-\nu^2)]$$
$$\Lambda = (\Omega^2-1)^{1/2}/\epsilon \gg 1$$
$$\lambda_3 = \{\Omega^2[(1-\nu^2)\Omega^2 - 1 - 3\nu] - 2\}/(\Omega^2-1)$$
$$n_3 = -\tfrac{1}{2} + [\lambda_3 + (\tfrac{1}{4})]^{1/2}$$
$$S(\phi) = \sin[(\Lambda^{1/2} + m\Lambda^{-1/2})\phi + (\pi/4)]$$
$$C(\phi) = \cos[(\Lambda^{1/2} + m\Lambda^{-1/2})\phi + (\pi/4)]$$
$$b = \Lambda^{1/2} - m\Lambda^{-1/2} \gg 1$$
$$m = \tfrac{1}{2}\{(\tfrac{1}{4}) + [\Omega^2(4+3\nu+\nu^2) - 2][2(\Omega^2-1)]^{-1}\}$$

The natural frequencies are found by solving the frequency condition (1). First approximations can be found by neglecting terms involving $\Lambda^{-1/2}$, in which case (1) becomes merely

$$[dP_{n_3}(\cos\phi)/d\phi]|_{\phi=\psi}S(\psi) = 0$$

The frequencies obtained by setting to zero the first factor,

$$dP_{n_3}(\cos\phi)/d\phi = 0 \quad \text{at} \quad \phi = \psi$$

are exactly the frequencies that would have been found by using membrane theory from the beginning. We shall focus attention on the lowest of these frequencies, which has (for $\nu = 0.3$)

$$n_3 = 0 \qquad \lambda_3 = 0 \qquad \Omega = [2/(1-\nu)]^{1/2} = 1.690 = \Omega_B$$

and often is described as the breathing frequency. If we now return to (1) and retain the terms involving $\Lambda^{-1/2}$, a simple trial-and-error calculation gives the more accurate value,

$$n_3 \approx -0.44 \qquad \Omega \approx 1.650$$

when $h/R = 0.05$ and $\psi = \pi/3$. This is in very close agreement with the value found for this case by Cohen[1]

$$\Omega = 1.646$$

The mode is found from (2) and (3), but it is necessary to replace these by (4) and (5) near $\phi = 0$. There are three terms involved in the expression for w, Eq. (2). First, the membrane contribution is $P_{n_3}(\cos\phi)$, which nearly is independent of ϕ. The remaining two terms contain the bending solutions. The more important of these involves $S(\phi)$ and is a fairly rapid, damped oscillation. The other term contains $e^{b(\phi-\psi)}$ and is negligible except near the edge, since $b \gg 1$ and $\phi \leq \psi$. This is an edge effect term, which allows us to satisfy all the boundary conditions at $\phi = \psi$. The u mode given by (3) contains three terms of the same type, but is much smaller than the w mode because the bending terms all are multiplied by the small quantity $\Lambda^{-1/2}$ and the membrane term involves the derivative of the nearly constant function $P_{n_3}(\cos\phi)$. The graph of these formulas is in good agreement with Fig. 1c of Cohen's[1] paper.

If the edge were clamped, the results of a pure membrane analysis would be the same as for the fixed-hinged case considered previously. We may expect, therefore, that a frequency near the membrane breathing frequency also will be found for a clamped edge. For this case, the frequency equation and modes can be found, as in the writer's[3] paper,‡

$$[dP_{n_3}(\cos\phi)/d\phi]_{\phi=\psi}[S(\psi) - C(\psi)][1 + \Lambda^{-1/2}(\tfrac{3}{8})\cot\psi] = 2\Lambda^{-1/2}P_{n_3}(\cos\psi)[\lambda_3 - (1-\nu) - (1-\nu^2)\Omega^2] \times [C(\psi) - \Lambda^{-1/2}(\tfrac{3}{8})\cot\psi S(\psi)] \quad (6)$$

‡ This frequency condition is slightly different from the one given as Eq. (55) of the paper[3] for reasons similar to the ones pertaining to the frequency equations for the fixed hinged case. Also, there is a minor error in Eq. (55).

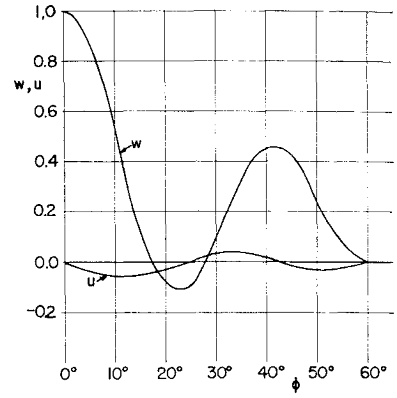

Fig. 1 Mode associated with the frequency $\Omega = 1.696$ for a clamped spherical shell with $h/R = 0.05$, $\psi = 60°$.

For $\phi \neq 0$,

$$w = P_{n_3}(\cos\phi) + P_{n_3}(\cos\psi)[S(\psi) - C(\psi)]^{-1}(\sin\psi/\sin\phi)^{1/2}\{-S(\phi) + [C(\psi) - (\tfrac{3}{8})\Lambda^{-1/2}\cot\psi S(\psi)]e^{b(\phi-\psi)}\} \quad (7)$$

$$u/(1+\nu) = [\lambda_3 - (1-\nu) - (1-\nu^2)\Omega^2]^{-1}dP_{n_3}(\cos\phi)/d\phi - \Lambda^{-1/2}P_{n_3}(\cos\psi)(\sin\psi/\sin\phi)^{1/2}[S(\psi) - C(\psi)]^{-1}\{C(\phi) + [C(\psi) - (\tfrac{3}{8})\Lambda^{-1/2}S(\psi)]e^{b(\phi-\psi)}\} \quad (8)$$

For $\phi \approx 0$,

$$w = P_{n_3}(\cos\phi) - \Lambda^{-1/4}P_{n_3}(\cos\psi) \times (\tfrac{1}{2}\pi\sin\psi)^{1/2}[S(\psi) - C(\psi)]^{-1}J_0(\Lambda^{1/2}\phi) \quad (9)$$

$$u/(1+\nu) = [\lambda_3 - (1-\nu) - (1-\nu^2)\Omega^2]^{-1} \times dP_{n_3}(\cos\phi)/d\phi + \Lambda^{-1/4}P_{n_3}(\cos\psi)(\tfrac{1}{2}\pi\sin\psi)^{1/2}[S(\psi) - C(\psi)]^{-1}J_1(\Lambda^{1/2}\phi) \quad (10)$$

The trial-and-error solution of (6) leads to

$$n_3 = 0.033 \qquad \lambda_3 = 0.0341 \qquad \Omega = 1.696$$

The modes given by (7–10) are plotted as Fig. 1. A frequency very close to this, $\Omega = 1.702$, was found for this case by D. R. Navaratna,[4] who employed a numerical scheme based on a variational procedure. The differences between his mode and that given by (7–10) are imperceptibly small on the scale of Fig. 1.

We may summarize by saying that for both clamped and fixed-hinged edges there is an extra natural frequency between the second and third ones given in Kalnins[2] Table 1. This extra frequency is quite close to the membrane breathing frequency. The mode is a superposition of membrane and bending terms which is considerably different from the pure membrane mode despite the closeness of the frequencies.

The frequency equations (1) and (6) show that the departure of this frequency from the pure membrane value $\Omega = \Omega_B$ usually is small, specifically, $0(\Lambda^{-1/2})$. Since this pure membrane frequency is independent of the edge angle ψ, we expect that any spherical dome executing axisymmetric vibrations will have a natural frequency near Ω_B if the edge is clamped or fixed-hinged.

The degree to which this expectation is fulfilled depends mostly on the size of $S(\psi)$ in the fixed-hinged case and $S(\psi) - C(\psi)$ in the clamped case. If $S(\psi) - C(\psi) \approx 0$ when $\Omega = \Omega_B$ in the clamped case, then the true natural frequency may differ considerably from Ω_B and similarly for the fixed-hinged case. For example, if $\psi = 56°$ and $h/R = 0.05$, then $S(\psi) - C(\psi) \approx 0$ in the clamped case for $\Omega = \Omega_B$. A trial-and-error calculation based on (6) gives, for $\nu = 0.3$, the value $\Omega \approx 1.79$ as compared with $\Omega = 1.696$ for $\psi = 60°$. The difference between Ω_B and the accurate value of Ω is much larger for the case $\psi = 56°$, where $S(\psi) - C(\psi) \approx 0$, than for $\psi = 60°$, where $S(\psi) - C(\psi) = 0(1)$.

The frequency conditions (1) and (6) support the conjecture that for a clamped or fixed-hinged edge the graph vs ψ of this frequency will consist of a rapid oscillation with fairly small amplitude (the amplitude depending roughly on $\epsilon^{1/2}$) about the value $\Omega = \Omega_B = [2/(1-\nu)]^{1/2}$.

References

[1] Cohen, G. A., "Computer analysis of asymmetric free vibrations of ring-stiffened orthotropic shells of revolution," AIAA J. **3**, 2305–2312 (1965).

[2] Kalnins, A., "Effect of bending on vibrations of spherical shells," J. Acoust. Soc. Am. **36**, 1355–1365 (1964).

[3] Ross, E. W., Jr., "Natural frequencies and mode shapes for axisymmetric vibration of deep spherical shells," J. App. Mech. **32**, 553–561 (1965).

[4] Navaratna, D. R., private communication, Aeroelastic & Structures Research Lab., Massachusetts Institute of Technology, Cambridge, Mass. (1965).

Reply by Author to E. W. Ross Jr.

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In reply to Ross' very pertinent comments, I would simply like to add the fact that a hand computation shows the ratio of bending strain energy to total strain energy for the third axisymmetric mode of the 60° fixed-hinged spherical shell ($h/R = 0.05$, $\nu = 0.3$) to be approximately 0.12. This result is in accordance with Ross' observation that although the third frequency is close to the pure membrane frequency, the mode is considerably different from the pure membrane mode.

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Errata: "Theory of Electrostatic Double Probe Comprised of Two Parallel Plates"

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[AIAA J. **4**, 442–450 (1966)]

Certain printout errors in the digital computer program pertaining to Figs. 2 through 6, have been brought to our attention.

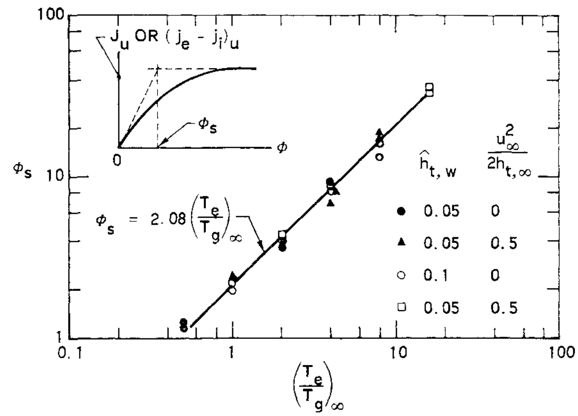

Fig. 8 Correlation of saturation potential.

According to the corrected numerical results, the assumption $(1/\beta_\infty)(\theta m/\hat{h})' \ll m'$ is more acceptable than the one on the second derivative made preceding Eq. (22). Equation (22) should be changed to

$$(1/Sc_i)[m + (1/\beta_\infty)(\theta m/\hat{h})]'' + f[m + (1/\beta_\infty)(\theta m/h)]' = 0$$

The solution of the preceding equation should replace that of Eq. (22) in Eqs. (24) and (39).

According to the revised correlation $(T_e/T_g)_\infty$ should be determined from the relationship given in Fig. 8 instead of Eq. (52) and Fig. 4. Also, $J_u = 0.5 + 0.47\,(T_e/T_g)_\infty$ should replace $J_u = 0.9$ in Eq. (53). The general discussions of the paper are still valid. A detailed discussion of the correction is given on the errata for Aerospace Corporation TDR-469(S5240-10)-3 under the same title.

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Errata: "Hypersonic Flow over a Delta Wing of Moderate Aspect Ratio"

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[AIAA J. **4**, 555–556 (1966)]

Incorrectly typeset expressions are:

1) The left-hand side of the equation for A_0 and B_0, which should read $A_0 = B_0$

2) The definition of G_M, which should read:

$$G_M \equiv -[(\gamma - 1)/2]Z_{Mc}$$

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